**Übungen Funktionen, Differenzieren und Integrieren – Lösungen**

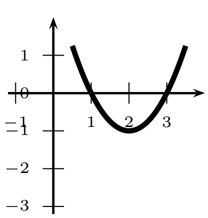
**Lösung von Aufgabe 1:**

$$\begin{aligned}
y &= x^2 - 4x + 3 \\
0 &= x^2 - 4x + 3 \Rightarrow x_{1/2} = 2 \pm \sqrt{4-3} \Rightarrow x_1 = 1, x_2 = 3 \\
y &= x^2 - 4x + 3 = (x-2)^2 - 1 \Rightarrow \text{Scheitelpunkt bei } S = (2, -1)
\end{aligned}$$

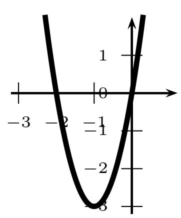
$$\begin{aligned}
y &= 3x^2 + 6x = 3x(x+2) \\
0 &= 3x^2 + 6x = 3x(x+2) \Rightarrow x_1 = 0, x_2 = -2 \\
y &= 3x^2 + 6x = 3(x+1)^2 - 3 \Rightarrow \text{Scheitelpunkt bei } S = (-1, -3)
\end{aligned}$$

$$\begin{aligned}
y &= 2x^2 - 10x + 12 \\
0 &= 2x^2 - 10x + 12 \\
y &= x^2 - 5x + 6 \Rightarrow x_{1/2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 6} = \frac{5}{2} \pm \sqrt{\frac{1}{4}} \\
&\Rightarrow x_1 = 2, x_2 = 3 \\
y &= 2x^2 - 10x + 12 = 2(x^2 - 5x + 6) = 2\left(x - \frac{5}{2}\right)^2 - \frac{1}{2} \Rightarrow \text{Scheitelpunkt bei } S = \left(\frac{5}{2}, -\frac{1}{2}\right)
\end{aligned}$$

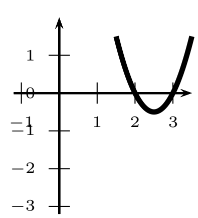
$$\begin{aligned}
y &= -x^2 + 8x - 16 \\
0 &= -x^2 + 8x - 16 \\
0 &= x^2 - 8x + 16 \Rightarrow x_{1/2} = 4 \pm \sqrt{16-16} \Rightarrow x_1 = x_2 = 4 \\
y &= -x^2 + 8x - 16 = -(x^2 - 8x + 16) = -(x-4)^2 \Rightarrow \text{Scheitelpunkt bei } S = (4, 0)
\end{aligned}$$



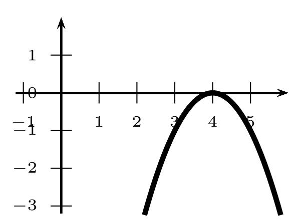
$y=x^2-4x+3$



$y=3x^2+6x$



$y=2x^2-10x+12$



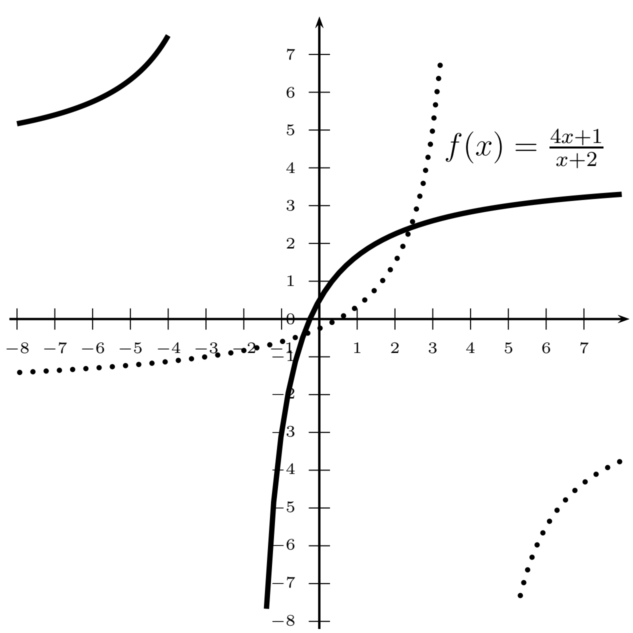
$y=-x^2+8x-16$

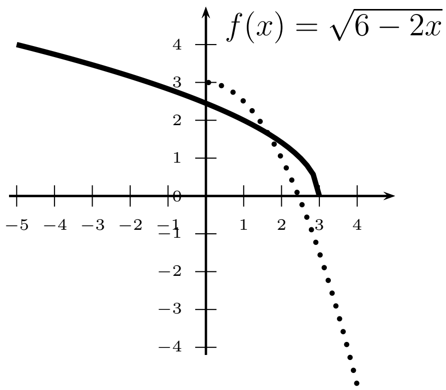
**Lösung von Aufgabe 2:**

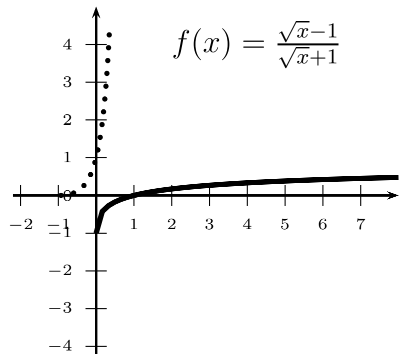
$$\begin{aligned}
f(x) = y &= \frac{4x+1}{x+2} \qquad D_f = \mathbb{R}\backslash\{-2\} \\
yx + 2y &= 4x + 1 \\
(y-4)x &= 1 - 2y \\
x &= \frac{1-2y}{y-4} \Rightarrow W_f = D_{f^{-1}}\mathbb{R}\backslash\{4\} \\
f^{-1}(x) &= \frac{1-2x}{x-4}
\end{aligned}$$

$$\begin{aligned}
f(x) = y &= \sqrt{6-2x} \qquad D_f = \{x| \quad x \le 3\}, \quad W_f = \{y| \quad y \ge 0\} \\
y^2 &= 6-2x \\
2x &= 6-y^2 \\
x &= 3-\frac{y^2}{2} \\
f^{-1}(x) &= 3-\frac{x^2}{2} \\
D_{f^{-1}} = W_f &= \{x| \quad x \ge 0\}, \quad \text{obwohl } 3-\tfrac{x^2}{2} \text{ für alle } x \text{ definiert ist}
\end{aligned}$$

$$\begin{aligned}
f(x) = y &= \frac{\sqrt{x}-1}{\sqrt{x}+1} \qquad D_f = \{x| \quad x \ge 0\} \\
y\sqrt{x}+y &= \sqrt{x}-1 \\
(y-1)\sqrt{x} &= -1-y \\
\sqrt{x} &= \frac{-1-y}{-1+y} \\
x &= \left(\frac{-1-y}{-1+y}\right)^2 = \left(\frac{1+y}{y-1}\right)^2 \\
f^{-1}(x) &= \left(\frac{1+x}{x-1}\right)^2 \\
\sqrt{x} &= \frac{-1-y}{-1+y} \ge 0 \Rightarrow \\
-1-y \ge 0 \quad \text{und} &\quad -1+y > 0 \Rightarrow \text{geht nicht} \\
-1-y \le 0 \quad \text{und} &\quad -1+y < 0 \Rightarrow -1 \le y < 1 \\
D_{f^{-1}} = W_f &= \{x| \quad -1 \le x < 1\}, \quad \text{obwohl } \left(\tfrac{1+y}{y-1}\right)^2 \text{ für alle } x \ne 1 \text{ definiert ist}
\end{aligned}$$






**Lösung von Aufgabe 3:**

$$y = e^{(x+1)^2} \qquad \begin{aligned} y &= \;= f(w) = e^w \\ w &= h(z) = z^2 \\ z &= g(x) = x+1 \end{aligned}$$

$$y = \left(e^{x+1}\right)^2 \qquad \begin{aligned} y &= f(w) = w^2 \\ w &= h(z) = e^z \\ z &= g(x) = x+1 \end{aligned}$$

$$y = \sin \ln(x+2) \qquad \begin{aligned} y &= f(w) = \sin w \\ w &= h(z) = \ln z \\ z &= g(x) = x+2 \end{aligned}$$

$$y = \frac{1}{\sin\sqrt{x}} \qquad \begin{aligned} y &= f(w) = \frac{1}{w} \\ w &= h(z) = \sin z \\ z &= g(x) = \sqrt{x} \end{aligned}$$

**Lösung von Aufgabe 4:**

$$y = \sqrt{5 - \tan\sqrt{x}} \qquad \begin{aligned} y &= f(v) = \sqrt{v} \\ v &= h(w) = 5 - w \\ w &= g(z) = \tan z \\ z &= k(x) = \sqrt{x} \end{aligned}$$

$$y = \log_3\left[\sqrt{2^x + 1}\right] \qquad \begin{aligned} y &= f(v) = \log_3 v \\ v &= h(w) = \sqrt{w} \\ w &= g(z) = z + 1 \\ z &= k(x) = 2^x \end{aligned}$$

$$y = \sin\left[\cos\frac{x-4}{3}\right]^2 \qquad \begin{aligned} y &= f(v) = \sin v \\ v &= h(w) = w^2 \\ w &= g(z) = \cos z \\ z &= k(x) = \frac{x-4}{3} \end{aligned}$$

**Lösung von Aufgabe 5:**
Kurvenpunkte mit Tangentenanstieg von 45° bedeutet: $y' = 1$.
Kurvenpunkte mit Tangentenanstieg von 135° bedeutet: $y' = -1$.

$$\begin{aligned} y = x^2 \quad &\Rightarrow \quad y' = 2x \\ y' = 1 \quad &\Longleftrightarrow \quad x = \frac{1}{2} \Rightarrow \text{Punkt } \left(\frac{1}{2}, \frac{1}{4}\right) \\ y' = -1 \quad &\Longleftrightarrow \quad x = -\frac{1}{2} \Rightarrow \text{Punkt } \left(-\frac{1}{2}, \frac{1}{4}\right) \end{aligned}$$

$$\begin{aligned} y = x^3 \quad &\Rightarrow \quad y' = 3x^2 \\ y' = 1 \quad &\Longleftrightarrow \quad x^2 = \frac{1}{3} \Longleftrightarrow x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{3}\sqrt{3} \Rightarrow \text{Punkte } \left(\frac{1}{3}\sqrt{3}, \frac{1}{9}\sqrt{3}\right); \left(-\frac{1}{3}\sqrt{3}, -\frac{1}{9}\sqrt{3}\right) \\ y' = -1 \quad &\Longleftrightarrow \quad x^2 = -\frac{1}{3} \quad \text{keine Punkte} \end{aligned}$$

$$
\begin{aligned}
y = \frac{x^2-3}{2x} \quad &\Rightarrow \quad y' = \frac{2x\cdot 2x - 2(x^2-3)}{4x^2} = \frac{x^2+3}{2x^2}\\
y' = 1 \quad &\Longleftrightarrow \quad \frac{x^2+3}{2x^2} = 1 \iff x^2 = 3 \iff x = \pm\sqrt{3} \Rightarrow \text{Punkte}\left(\sqrt{3},0\right);\left(-\sqrt{3},0\right)\\
y' = -1 \quad &\Longleftrightarrow \quad \frac{x^2+3}{2x^2} = -1 \iff x^2 = -3 \quad \text{keine Punkte}\\
y = \frac{x^2+x+14}{x+2} \quad &\Rightarrow \quad y' = \frac{(2x+1)(x+2) - 1\cdot(x^2+x+14)}{(x+2)^2} = \frac{x^2+4x-12}{x^2+4x+4}\\
y' = 1 \quad &\Longleftrightarrow \quad \frac{x^2+4x-12}{x^2+4x+4} = 1 \iff -12 = 4 \quad \text{keine Punkte}\\
y' = -1 \quad &\Longleftrightarrow \quad \frac{x^2+4x-12}{x^2+4x+4} = -1 \iff 2x^2+8x-8 = 0 \iff x = 2\pm\sqrt{8} = 2\pm 2\sqrt{2}\\
&\quad \text{Punkte } (0,828; 5,485), \quad (-4,828; -11,485)
\end{aligned}
$$

**Lösung von Aufgabe 6:**
Winkel der Tangente sei $\alpha$:

$$
\begin{aligned}
y = \sqrt{x} \quad &\Rightarrow \quad y' = \frac{1}{2\sqrt{x}}, \Rightarrow y'(x_0=1) = \frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26,57^\circ\\
y = \sqrt[3]{x+1} \quad &\Rightarrow \quad y' = \frac{1}{3}(x+1)^{-\frac{2}{3}}, \Rightarrow y'(x_0=0) = \frac{1}{3} \Rightarrow \tan\alpha = \frac{1}{3} \Rightarrow \alpha = 18,43^\circ\\
y = x\sqrt{x+1} \quad &\Rightarrow \quad y' = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}},\\
&\Rightarrow \quad y'(x_0=3) = 2 + \frac{3}{4} = 2,75 \Rightarrow \tan\alpha = 2,75 \Rightarrow \alpha = 70^\circ
\end{aligned}
$$

**Lösung von Aufgabe 7:**
Tangente hat den Anstieg $f'(x_0)$ also $m = f'(x_0)$.
Tangente und Funktion berühren sich:

$$
f(x_0) = m\cdot x_0 + n = f'(x_0)\cdot x_0 + n \Rightarrow n = f(x_0) - f'(x_0)\cdot x_0
$$

$$
\Rightarrow y = mx + n = f'(x_0)\cdot x + f(x_0) - f'(x_0)\cdot x_0 = f'(x_0)(x-x_0) + f(x_0)
$$

$$
\begin{aligned}
f(x) = x^2+1 \quad &\Rightarrow \quad f'(x) = 2x\\
P_1(1,y_1) \quad &\Rightarrow \quad f'(1) = 2, f(1) = 2 \Rightarrow \text{Tangente } y = 2(x-1)+2 = 2x\\
P_2(-1,y_2) \quad &\Rightarrow \quad f'(-1) = -2, f(1) = 2 \Rightarrow \text{Tangente } y = -2(x+1)+2 = -2x
\end{aligned}
$$

$$
\begin{aligned}
f(x) = x^3-3x^2+x+1 \quad &\Rightarrow \quad f'(x) = 3x^2-6x+1\\
P_1(0,y_1) \quad &\Rightarrow \quad f'(0) = 1, f(0) = 1 \Rightarrow \text{Tangente } y = 1(x-0)+1 = x+1\\
P_2(2,y_2) \quad &\Rightarrow \quad f'(2) = 1, f(2) = -1 \Rightarrow \text{Tangente } y = 5(x-2)-1 = x-3
\end{aligned}
$$

**Lösung von Aufgabe 8:**

$$\begin{aligned}
y &= x\sqrt{9x-x^2} \Rightarrow x \in [0,9] \\
y' &= \sqrt{9x-x^2} + \frac{x(9-2x)}{2\sqrt{9x-x^2}} = \frac{2(9x-x^2)+x(9-2x)}{2\sqrt{9x-x^2}} = \frac{27x-4x^2}{2\sqrt{9x-x^2}} \\
y'=0 &\Rightarrow 27x-4x^2=0 \Rightarrow x_1=0, x_2=\frac{27}{4}
\end{aligned}$$

Achtung: Für $x=0$ ist Nenner $=0$, daher nicht definiert

$$\begin{aligned}
y &= x^2\sqrt{25-x^2} \Rightarrow x \in [-5,5] \\
y' &= 2x\sqrt{25-x^2} + x^2 \cdot \frac{1}{2\sqrt{25-x^2}} \cdot (-2x) \\
&= 2x\sqrt{25-x^2} - \frac{x^3}{\sqrt{25-x^2}} = \frac{2x(25-x^2)-x^3}{\sqrt{25-x^2}} = \frac{50x-3x^3}{\sqrt{25-x^2}} \\
y'=0 &\Rightarrow 50x-3x^3=0 \Rightarrow x_1=0, \quad x_{2/3}=\pm\sqrt{\frac{50}{3}}
\end{aligned}$$

Achtung: $x_{2/3}=\pm\sqrt{\frac{50}{3}}$ sind nicht im Definitionsbereich

$$\begin{aligned}
y &= \sqrt{\frac{x}{2-x}} \Rightarrow x \in [0,2) \\
y' &= \frac{1}{2\sqrt{\frac{x}{2-x}}} \cdot \left(\frac{2-x+x}{(2-x)^2}\right) = \sqrt{\frac{2-x}{x}} \cdot \frac{1}{(2-x)^2} \\
y' &= 0 \Rightarrow 2-x=0 \Rightarrow 2. \text{ Faktor nicht def., also keine Nullstellen}
\end{aligned}$$

**Lösung von Aufgabe 9:**

$$\begin{aligned}
\int (x^3-5x^2+7x-2)\,dx &= \frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 2x + C \\
\int \frac{2}{x^3}\,dx &= \int 2x^{-3}\,dx = -x^{-2}+C \\
\int \frac{1}{3}\sqrt[3]{x}\,dx &= \int \frac{1}{3}x^{\frac{1}{3}}\,dx = \frac{1}{4}x^{\frac{4}{3}} + C
\end{aligned}$$

**Lösung von Aufgabe 10:**

$$\begin{aligned}
\int \sqrt{x\cdot\sqrt[3]{x}}\,dx &= \int x^{\frac{1}{2}} \cdot x^{\frac{1}{6}}\,dx = \int x^{\frac{2}{3}}\,dx = \frac{3}{5}x^{\frac{5}{3}} + C \\
\int x\sqrt{x}\,dx &= \int x\cdot x^{\frac{1}{2}}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + C \\
\int \frac{2+\sqrt{x}}{x^2}\,dx &= \int \left(\frac{2}{x^2}+\frac{\sqrt{x}}{x^2}\right)dx = \int \left(2x^{-2}+x^{-\frac{3}{2}}\right)dx \\
&= -2x^{-1} - 2x^{-\frac{1}{2}} + C = -\left(\frac{2}{x}+\frac{2}{\sqrt{x}}\right) + C
\end{aligned}$$

**Lösung von Aufgabe 11:**

Achtung: $\frac{1}{2}x^3$ hat zwischen $-2$ und $2$ eine Nullstelle.

$$\begin{aligned}
A &= \; = \left|\int\limits_{-2}^{0} \frac{1}{2}x^3 \; dx\right| + \int\limits_{0}^{2} \frac{1}{2}x^3 \; dx \\
&= \left|\left[\frac{1}{8}x^4\right]_{-2}^{0}\right| + \left[\frac{1}{8}x^4\right]_{0}^{2} = 2 + 2 = 4 \\
A &= \int_{-2}^{2}\left(\frac{x^2}{4} - 1\right)^2 dx = \int_{-2}^{2}\left(\frac{x^4}{16} - \frac{x^2}{2} + 1\right) dx \\
&= \left[\frac{x^5}{80} - \frac{x^3}{6} + x\right]_{-2}^{2} = \frac{32}{80} - \frac{8}{6} + 2 + \frac{32}{80} - \frac{8}{6} + 2 = \frac{32}{15} \\
A &= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx = \Big[\sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2
\end{aligned}$$

**Lösung von Aufgabe 12:**

$$\begin{aligned}
3 - \frac{1}{2}x^4 &= 3 - 4x \\
-\frac{1}{2}x^4 &= -4x \iff x_1 = 0, x_2 = 2 \\
A &= \int\limits_{0}^{2}\left(3 - \frac{1}{2}x^4 - 3 + 4x\right) \; dx = \int\limits_{0}^{2}\left(4x - \frac{1}{2}x^4\right) \; dx \\
&= \left[2x^2 - \frac{1}{10}x^5\right]_{0}^{2} = 8 - 3{,}2 = 4{,}8
\end{aligned}$$

$$\begin{aligned}
x^3 + 7 &= x^3 - x^2 + 3x + 5 \\
0 &= x^2 - 3x + 2 \\
x_{1/2} &= \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} \\
x_1 &= 2, \quad x_2 = 1 \\
A &= \int\limits_{1}^{2}(x^3 - x^2 + 3x + 5 - x^3 - 7) \; dx = \int\limits_{1}^{2}(-x^2 + 3x - 2) \; dx \\
&= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x\right]_{1}^{2} = -\frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2 = \frac{1}{6}
\end{aligned}$$